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On the Solution of Certain Types of Linear Differential Equations in Infinitely Many Variables.

BY WEBSTER G. SIMON.

The main purpose of this paper is to prove the existence of certain types of solutions of particular kinds of linear differential equations with periodic coefficients in infinitely many variables. As a means to this end the existence of exponential solutions is established for certain types of linear differential equations in infinitely many variables with constant coefficients.

The starting point is the existence theorem given by von Koch,* and a generalization of Poincaré's theorem † concerning the development of the solutions of the differential equations as power series in a parameter μ when the functions appearing in the differential equations are themselves power series in μ . Then our work is very similar to that of the finite case, the finite determinants becoming infinite determinants which together with all their first minors converge absolutely.

The type of determinant used in this paper is more general than the normal determinant, and is the following.‡ The determinant

$$\begin{vmatrix} 1 + a_{11}, & a_{12}, & a_{13}, & \cdots \\ a_{21}, & 1 + a_{22}, & a_{23}, & \cdots \\ a_{31}, & a_{32}, & 1 + a_{33}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots \end{vmatrix}$$

is such that there exist two sets of positive constants $S_1, S_2, \cdots, T_1, T_2, \cdots$, which are of such a nature that $|a_{ij}| < S_i T_j$, and $\sum_{i=1}^{\infty} S_i T_i$ converges. This is a type of infinite determinant given by von Koch,§ which together with all its minors converges absolutely.

Using the methods thus indicated, we find that many of the phenomena of the finite systems are carried over into the infinite systems of differential equations.||

* Von Koch, *Öfversigt af Kongliga Vetenskaps Akademiens Förhandlingar*, Vol. 56 (1899), pp. 395-411.

† Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, chapter II.

§ Von Koch, *Acta Mathematica*, Vol. 24 (1901), pp. 89-122. Hereafter this paper will be denoted by K.

|| See Moulton and MacMillan, *AMERICAN JOURNAL OF MATHEMATICS*, Vol. 33 (1911), pp. 63-96.

To prove the convergence of (2), consider the system of differential equations

$$(5) \quad \xi'_i = S_i \sum_{j=1}^{\infty} T_j \xi_j, \quad i = 1, 2, \dots \infty.$$

Upon substituting $\xi_i = \beta_i + \sum_{j=1}^{\infty} \gamma_i^{(j)} t^j$ in (5) and equating coefficients, we get

$$(6) \quad \begin{cases} \gamma_i^{(1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(0)}, \\ 2\gamma_i^{(2)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(1)}, \\ \dots \dots \dots \\ (\lambda + 1)\gamma_i^{(\lambda+1)} = S_i \sum_{j=1}^{\infty} T_j \gamma_j^{(\lambda)}, \text{ where } \gamma_i^{(0)} = \beta_i. \end{cases}$$

On comparison of (6) and (4) we see that $\gamma_i^{(j)} > |\beta_i^{(j)}|$, $i, j = 1, 2, \dots \infty$. Hence ξ_i dominates x_i for every i .

From (5) we have

$$(7) \quad \frac{1}{S_1} \xi'_1 = \frac{1}{S_2} \xi'_2 = \dots = \xi'.$$

On taking $\xi(0) = 0$, we have

$$(8) \quad \xi_i - \beta_i = S_i \xi, \quad i = 1, 2, \dots \infty.$$

Therefore each equation (5) reduces to

$$(9) \quad \xi' = C\xi + K,$$

where $C = S_1 T_1 + S_2 T_2 + \dots$, $K = \beta_1 T_1 + \beta_2 T_2 + \dots$.

From the theory of a finite system of differential equations we know that (9) has a unique analytic solution for $|t| < R$. On combination of this fact with (8), we see that the solution (2) of (1) converges when $|t| < R$.

Now we define as a fundamental set of solutions of (1), a set such that every solution of (1) can be expressed as linear homogeneous functions with constant coefficients of the elements of the set. Then if we denote by ϕ_{ij} the elements of the fundamental set, it follows that the determinant of the ϕ_{ij} converges and is not zero for all $|t| < R$, and conversely. Furthermore it

has the form $\Delta = \Delta_0 e^{\int_{t_0}^t \sum_{i=1}^{\infty} \theta_{ii}(t) dt}$, for all $|t| < R$, where Δ_0 is the value of the determinant Δ when $t = t_0$. If, for example, we take the solutions defined by $\phi_{ii}(0) = 1$ and $\phi_{ij}(0) = 0$, $i \neq j$, we see from (8) and (9) that

$$|\phi_{ii}| \leq 1 + \frac{S_i T_i}{C} |e^{Ct} - 1|, \text{ and } |\phi_{ij}| \leq \frac{S_i T_j}{C} |e^{Ct} - 1|, i \neq j.$$

We see that the determinant of the ϕ_{ij} is absolutely convergent.

Theorem II. *Suppose that the system (1) satisfies the following hypotheses:*

(H₁) *The θ_{ij} are expansible as power series in a parameter μ , which converge for all real $|t| < R$ if $|\mu| < \rho$.*

(H₂) *For $\mu = 0$, the $\theta_{ij} = a_{ij}$, where the a_{ij} are constants.*

(H₃) *The θ_{ij} satisfy all the hypotheses of Theorem I uniformly with respect to μ if $|\mu| < \rho$.*

Then the solutions of (1) can be expressed as power series in μ , which converge for $|\mu| \leq \rho_0 < \rho$ and for all real $|t| < R$.

In Theorem I we showed that $x_i - \beta_i = \sum_{j=1}^{\infty} \beta_i^{(j)} t^j$, $|t| < R$, $|\mu| < \rho$, where $n\beta_i^{(n)} = P_i^{(n)}(\beta_j^{(0)}, \dots, \beta_j^{(n-1)})$.

In particular $\beta_i^{(1)} = \sum_{j=1}^{\infty} a_{ij}^{(0)} \beta_j^{(0)}$. The $a_{ij}^{(0)}$ are power series in μ . Hence $\beta_j^{(0)} a_{ij}^{(0)} = \sum_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$, $\beta_i^{(1)} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \beta_{ijk}^{(0)} \mu^k$ which converge for $|\mu| < \rho$. Then by a well-known theorem in the theory of double series, we infer that $\beta_i^{(1)} = \sum_{k=1}^{\infty} \beta_{ik}^{(1)} \mu^k$, $|\mu| < \rho$. In a similar fashion we proceed step by step, and show that the $\beta_i^{(n)}$ are also power series in μ , which converge for $|\mu| < \rho$. So we have

$$(10) \quad x_i = \sum_{k=1}^{\infty} \phi_{ik}(\mu), \quad i = 1, 2, \dots, \infty,$$

which is uniformly convergent for $|\mu| \leq \rho_0 < \rho$, and where $\phi_{ik}(\mu) = \phi_{ik}^{(0)} + \phi_{ik}^{(1)} \mu + \dots$. Then by a well-known theorem of Weierstrass, we know that we can rearrange the series (10), and write it in the form

$$x_i = \sum_{j=1}^{\infty} x_i^{(j)}(t) \mu^j, \quad |\mu| \leq \rho_0 < \rho, \quad |t| < R, \quad i = 1, 2, \dots, \infty.$$

II

Now let us assume that the θ_{ij} of Theorem I are constants; and let these constants be denoted by a_{ij} . We inquire whether the system (1) has a solution of the form

$$(11) \quad x_i = c_i e^{at},$$

where by (H₃) of Theorem I $\sum_{j=1}^{\infty} a_{ij} c_j$ converges. Necessary and sufficient con-

Theorem III: *If the θ_{ij} of (1) are constants and if λ_0 is a root of $\Delta(\lambda) = 0$ such that all minors of order $r-1$, but not all of order r , vanish for $\lambda = \lambda_0$, then there exist r independent solutions of (1) of the form (11).*

Now the characteristic equation, since it is an integral transcendental function, may have no finite roots, it may have a finite number of finite roots, or a denumerably infinite number of roots. In the first case the c_i are all zero, in the second case, a finite number of solutions of the form (11) exists and in the last case an infinite number of the form (11) exists. This is the case we shall study, for this infinite set constitutes a fundamental set of solutions as the following discussion shows.

For each λ that is a root of $\Delta(\lambda) = 0$ there is a solution of (1) of the form

$$(16) \quad x_{ij} = c_{ij} e^{a_j t}, \quad i, j = 1, 2, \dots \infty.$$

We know that the system (1) admits the fundamental set of solutions ϕ_{ij} , $i, j = 1, 2, \dots \infty$, where $\phi_{ii}(0) = 1$, $\phi_{ij}(0) = 0$, $i \neq j$. Consequently

$$x_{i1} = \sum_{k=1}^{\infty} A_k^{(1)} \phi_{ik}, \quad i = 1, 2, \dots \infty.$$

First let us assume that the a_j are all distinct. Then it follows immediately that the system (16) constitutes a fundamental set. For consider x_{i1} , ϕ_{ij} , $i = 1, 2, \dots \infty$, $j = 2, 3, \dots \infty$.

$$\begin{vmatrix} \sum_{k=1}^{\infty} A_k^{(1)} \phi_{1k}, & \phi_{12}, & \dots \\ \sum_{k=1}^{\infty} A_k^{(1)} \phi_{2k}, & \phi_{22}, & \dots \\ \dots & \dots & \dots \end{vmatrix} = A_1^{(1)} \begin{vmatrix} \phi_{11}, & \phi_{12}, & \dots \\ \phi_{21}, & \phi_{22}, & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

Choose the initial conditions such that $A_1^{(1)} = 1$, which is no actual restriction since we are merely determining the arbitrary constant which enters, and we can multiply our solution by a constant, not zero, when we are through if we choose. Thus x_{i1} , ϕ_{ij} , $i = 1, 2, \dots \infty$, $j = 2, 3, \dots \infty$, constitute a fundamental set. Similarly we show that x_{i1} , x_{i2} , ϕ_{ij} , constitute a fundamental set, where $j = 3, 4, \dots \infty$. Continuing in this manner and passing to the limit we see that the x_{ij} , $i, j = 1, 2, \dots \infty$, constitute a fundamental set. A result of this determination of the arbitrary constant is that $c_{ii} = 1$.

Next let $a_2 = a_1$, and $a_j \neq a_1$, $j = 3, 4, \dots \infty$. There are two cases that may arise here, viz., all of the first minors of $\Delta(\lambda)$ vanish when $a = a_1$, or not all vanish. Consider the former case. Then (15) assures us that we can solve for the c_{ij} in terms of two of them. On choosing the notation so

that $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ is not zero, we get two independent solutions of the form

$$x_{i1} = c_{i1}e^{a_1 t}, \quad x_{i2} = c_{i2}e^{a_1 t}, \quad i = 1, 2, \dots, \infty,$$

which with ϕ_{ij} , $j = 3, 4, \dots, \infty$, constitute a fundamental set; for

$$\begin{vmatrix} x_{11}, & x_{12}, & \phi_{13}, & \dots \\ x_{21}, & x_{22}, & \phi_{23}, & \dots \\ \dots, & \dots, & \dots, & \dots \end{vmatrix} = \begin{vmatrix} A_1^{(1)}\phi_{11} + A_2^{(1)}\phi_{12}, & A_1^{(2)}\phi_{11} + A_2^{(2)}\phi_{12}, & \phi_{13}, & \dots \\ A_1^{(1)}\phi_{21} + A_2^{(1)}\phi_{22}, & A_1^{(2)}\phi_{21} + A_2^{(2)}\phi_{22}, & \phi_{23}, & \dots \\ \dots, & \dots, & \dots, & \dots \end{vmatrix}$$

and the $A_2^{(1)}, A_1^{(2)}$ may be taken to be zero, and $A_1^{(1)} = A_2^{(2)} = 1$. So these solutions, we see, constitute a fundamental set.

Next let us assume that not all the first minors of $\Delta(\lambda)$ vanish when $a = a_2 = a_i$; and let us choose the notation so that the first minor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not zero. Then to get a solution associated with a_1 , we make the transformation $y = b_1x_1 + b_2x_2 + \dots$, $y' = b_1x'_1 + b_2x'_2 + \dots$ and, if possible, determine the b 's in such a manner that $y' = ay$. On substituting these equations in (1) we get $b_1[a_{11}x_1 + a_{12}x_2 + \dots] + b_2[a_{21}x_1 + a_{22}x_2 + \dots] + \dots = a[b_1x_1 + b_2x_2 + \dots]$.

This equation must hold for all initial values of the x 's. Therefore it is an identity in them; and the b 's satisfy the following set of equations

$$(17) \quad \begin{cases} (a_{11} - a)b_1 + a_{21}b_2 + \dots = 0, \\ a_{12}b_1 + (a_{22} - a)b_2 + \dots = 0, \\ \dots \dots \dots \end{cases}$$

In order that these equations have a solution for the b 's not all zero, we get as a necessary and sufficient condition, just as in the case of c 's in (13), $\Delta(\lambda) = 0$. This condition is satisfied, for we have assumed that when $a = a_1$, the characteristic equation is satisfied. Furthermore we have chosen the notation such that the minor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not zero when $a = a_1$. Then we get for the b 's

$$b_j = m_j b_1, \quad j = 1, 2, \dots, \infty,$$

where $m_j = \frac{\begin{pmatrix} j \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$. For convenience we shall take $b_1 = 1$. Then the equations become

$$(1') \quad \begin{cases} y'_1 = a_1 y_1, \\ x'_2 = a_{21}y_1 + (a_{22} - a_{21}m_2)x_2 + (a_{23} - a_{21}m_3)x_3 + \dots, \\ \dots \dots \dots \\ x'_n = a_{n1}y_1 + (a_{n2} - a_{n1}m_2)x_2 + (a_{n3} - a_{n1}m_3)x_3 + \dots, \\ \dots \dots \dots \end{cases}$$

Then by a precisely similar argument to the foregoing we can show that

$$x_{i1}, x_{i2}, \phi_{ij}, i = 1, 2, \dots \infty, j = 3, 4, \dots \infty,$$

constitute a fundamental set.

When for a given value of α the characteristic equation has a root of higher multiplicity, discussions similar to the foregoing must be made.

We should expect to treat next the case that $\Delta(\lambda) = 0$ has a root of infinite multiplicity. But such a case cannot arise. For we have seen that $\Delta(\lambda)$ is an integral transcendental function of λ . Therefore it can be expanded in a Taylor's series in the neighborhood of any finite λ . Suppose that λ_0 were a finite root of infinite multiplicity. Then $\Delta(\lambda)$ vanishes together with all of its derivatives when $\lambda = \lambda_0$. Hence $\Delta(\lambda)$ vanishes at every point in the neighborhood of λ_0 . Then it is identically zero. But this is not true, for when $\lambda = 0$, $\Delta(\lambda) = 1$. Hence $\Delta(\lambda)$ has no root of infinite multiplicity.

If $\alpha = 0$, we can not make the transformation which carries the system (12) into the system (13), and the determinant of (12) diverges. Then we do not know whether the equations (12) have a solution for the c_j not all zero; but each special case must be considered as it arises.

To sum up our results we have

Theorem IV. *If and only if $\Delta(\lambda)$ has an infinite number of roots λ_i , the system (1) has a fundamental set of solutions each of whose elements is of the form*

$$x_{ij} = e^{\alpha_j t} \psi_{ij}(t), i, j = 1, 2, \dots \infty,$$

where $\alpha_i = -1/\lambda_i$ and the ψ_{ij} are polynomials in t of degree at most $(n - 1)$, and n is the order of multiplicity of the root λ_j .

III

Now we shall assume that the θ_{ij} of Theorem I are periodic with the period 2π . We have seen that the system (1) has, as a fundamental set of solutions, the set $\phi_{ij}(t)$, where

$$(18) \quad \phi_{ii}(0) = 1, \phi_{ij}(0) = 0, i \neq j, i, j = 1, 2, \dots \infty.$$

Let us make the transformation

$$(19) \quad x_i = e^{\alpha t} y_i,$$

where α is an undetermined constant. Then the equations (1) become

$$(20) \quad y'_i + \alpha y_i = \sum_{j=1}^{\infty} \theta_{ij}(t) y_j, i = 1, 2, \dots \infty.$$

infinite number of finite roots, the system (1) has a fundamental set of solutions, each of the elements of which is of the form (19).

The discussion of the form of the solutions follows precisely the same lines as that in the case of constant coefficients except when σ_0 is an n -fold root of (26) and not all the $(n - k)$ th, $k > 0$, minors of $D(\sigma)$ vanish for $\sigma = \sigma_0$.

Now let us assume that $\sigma = \sigma_0$ is an n -fold root of (26) and not all the first minors of $D(\sigma)$ vanish. Then there is only one solution of (1) of the form $x_i = e^{a_1 t} y_i$, where the y_i are expressed as in (21). And let us choose the notation so that a minor corresponding to the elements of the first column is not zero.

Then we take as a new set of solutions

$$(27) \quad x_{i1} = e^{a_1 t} y_{i1}, \quad x_{ij} = \phi_{ij}(t), \quad i = 1, 2, \dots \infty, \quad j = 2, 3, \dots \infty.$$

This set of solutions can be shown to constitute a fundamental set in a precisely similar fashion to that in II. Then we make the transformation

$$x_i = e^{a t} z_i, \quad i = 1, 2, \dots \infty.$$

As above we get

$$(28) \quad z_i = e^{-a t} [A_1 e^{a_1 t} y_{i1} + \sum_{j=2}^{\infty} A_j \phi_{ij}(t)], \quad i = 1, 2, \dots \infty.$$

Necessary and sufficient conditions that the z_i be periodic with the period 2π are

$$z_i(2\pi) - z_i(0) = 0, \quad i = 1, 2, \dots \infty.$$

On imposing these conditions on (28), we get

$$(29) \quad A_1 [e^{-2(a-a_1)\pi} y_{i1}(2\pi) - y_{i1}(0)] + \sum_{j=2}^{\infty} A_j [\phi_{ij}(2\pi) e^{-2a\pi} - \phi_{ij}(0)] = 0.$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the equations (29) become

$$(30) \quad A_1 (1 - \sigma/\sigma_1) y_{i1}(0) + \sum_{j=2}^{\infty} A_j \sigma \phi_{ij} + A_k [1 + \sigma(\phi_{kk} - 1)] = 0, \\ k \neq j, \quad k = 2, 3, \dots \infty, \quad i = 1, 2, \dots \infty.$$

The fundamental equation for the equations (30) is

$$(31) \quad \begin{vmatrix} (1 - \sigma/\sigma_1) y_{11}(0), & \sigma \phi_{12}, & \dots \\ (1 - \sigma/\sigma_1) y_{21}(0), & 1 + \sigma(\phi_{22} - 1), & \dots \\ \dots & \dots & \dots \end{vmatrix} = 0.$$

Making use of (21) and taking $A_1 = 1$, equation (31) becomes

$$\begin{vmatrix} \sigma \sum_{j=1}^{\infty} A_j \phi_{1j} - (\sigma - 1), & \sigma \phi_{12} & \cdots \\ \sigma \sum_{j=1}^{\infty} A_j \phi_{2j} - A_2(\sigma - 1), & 1 + \sigma(\phi_{22} - 1), & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} = D(\sigma) = 0.$$

So we have

$$(32) \quad D(\sigma) = (1 - \sigma/\sigma_1) D_1 = (1 - \sigma/\sigma_1) \begin{vmatrix} y_{11}(0), & \sigma \phi_{12}, & \cdots \\ y_{21}(0), & 1 + \sigma(\phi_{22} - 1), & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Since $\sigma = \sigma_1$ is an n -fold root of $D(\sigma) = 0$, $D_1(\sigma)$ has the factor $(1 - \sigma/\sigma_1)^{n-1}$. Since (27) constitutes a fundamental set, any solution can be expressed in the form

$$(33) \quad x_i = B_1 e^{a_1 t} y_{i1} + \sum_{j=2}^{\infty} B_j \phi_{1j}, \quad i = 1, 2, \cdots, \infty.$$

Now let us make the transformation, corresponding to (19), to get a second solution associated with a_1 .

$$(34) \quad x_{i2} = e^{a_1 t} (y_{i2} + t y_{i1}), \quad i = 1, 2, \cdots, \infty.$$

On imposing the condition that x_{i2} shall satisfy the system (1), we find since $e^{a_1 t} y_{i1}$ is a solution,

$$(35) \quad y'_{i2} + a_1 y_{i2} = \sum_{j=2}^{\infty} \theta_{ij}(t) y_{j2} - y_{i1}, \quad i = 1, 2, \cdots, \infty.$$

From the form of (35) we see that sufficient conditions that the y_{i2} shall be periodic with the period 2π are

$$(36) \quad y_{i2}(2\pi) - y_{i2}(0) = 0 = \sum_{j=2}^{\infty} B_j [e^{-2a_1 \pi} \phi_{ij}(2\pi) - \phi_{ij}(0)] \\ - 2\pi y_{i1}(0) = 0.$$

On substituting $e^{-2a_1 \pi} = \sigma_1/\sigma_1 - 1$, (36) becomes

$$(37) \quad -2\pi(\sigma_1 - 1)y_{i1}(0) + \sum_{j=2}^{\infty} B_j \phi_{ij} + B_k [1 + \sigma_1(\phi_{kk} - 1)] = 0, \quad k \neq j.$$

The condition that these equations be consistent is, since $\sigma_1 \neq 1$,

$$D_1(\sigma) = \begin{vmatrix} y_{11}(0), & \sigma_1 \phi_{12}, & \cdots \\ y_{21}(0), & 1 + \sigma_1(\phi_{22} - 1), & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

In (32) we showed that D_1 vanishes $(n-1)$ times when $\sigma = \sigma_1$. By hypotheses not all the first minors corresponding to the elements of the first column

are zero for $\sigma = \sigma_1$. Hence we can solve equations (37) for B_2, B_3, \dots , terms of $y_{i_1}(0)$. Consequently in this case we get a second solution associated with α_1 , which is of the form (34).

In a similar manner we can go ahead step by step and get the following group of solutions associated with α_1

$$\begin{cases} x_{i_1} = e^{\alpha_1 t} y_{i_1}, \\ x_{i_2} = e^{\alpha_1 t} [y_{i_2} + t y_{i_1}], \\ \vdots \\ x_{i_n} = e^{\alpha_1 t} [y_{i_n} + t y_{i_{n-1}} + \dots + 1/(n-1)! t^{n-1} y_{i_1}]. \end{cases}$$

If σ_1 is a triple root of $D(\sigma) = 0$, such that all first minors are zero, but not all the second minors are zero, the solutions associated with α_1 are

$$x_{i_1} = e^{\alpha_1 t} y_{i_1}, x_{i_2} = e^{\alpha_1 t} y_{i_2}, x_{i_3} = e^{\alpha_1 t} [y_{i_3} + t(y_{i_1} + y_{i_2})].$$

All the sub-cases can be treated, as they arise, by the methods given here.

IV

Now let us assume that the system of differential equations (1) satisfy the hypotheses of the second existence theorem, and in addition the coefficients in the power series expansions of the θ_{ij} are separately periodic with the period 2π . That theorem tells us that the solutions of (1) can be written in the form

$$(39) \quad x_{ij} = \sum_{k=1}^{\infty} x_{ij}^{(k)}(t) \mu^k, \quad i, j = 1, 2, \dots \infty.$$

And we shall take the initial conditions such that

$$(40) \quad x_{ij}(0) = \sum_{k=1}^{\infty} x_{ij}^{(k)}(0) \mu^k \equiv c_{ij},$$

whence

$$x_{ij}^{(0)}(0) = c_{ij}, \quad x_{ij}^{(k)}(0) = 0, \quad k = 1, 2, \dots \infty.$$

where the c_{ij} are constants such that $c_{ii} = 1$, and their determinant is absolutely convergent. These conditions coupled with the fact that if the determinant of a set of solutions converges and is not zero when $t = 0$, it converges and is not zero for every value of t for which $\sum_{i=1}^{\infty} \theta_{ii}$ converges, show that the system (40) constitutes a fundamental set of solutions.

Now we inquire if we can find solutions of (1) of the form

$$(41) \quad x_i = e^{\alpha t} y_i,$$

where the y_i are periodic with the period 2π , and α is a constant which remains

to be determined. After making the transformation (41), the differential equations and their solutions become

$$(42) \begin{cases} y'_1 + \alpha y_i = \sum_{j=1}^{\infty} [a_{ij} + \sum_{k=1}^{\infty} \theta_{ij}^{(k)} \mu^k] y_j, & i = 1, 2, \dots \infty, \\ y_i = \sum_{j=1}^{\infty} A_j e^{-\alpha t} [x_{ij}^{(0)}(t) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(t) \mu^k], & i = 1, 2, \dots \infty. \end{cases}$$

On imposing the conditions that the y_i be periodic with the period 2π , viz., $y_i(2\pi) - y_i(0) = 0$, we get

$$(43) \quad 0 = \sum_{j=1}^{\infty} A_j [e^{-2\alpha\pi} x_{ij}^{(0)}(2\pi) - c_{ij} + e^{-2\alpha\pi} \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k], \quad i = 1, 2, \dots \infty.$$

After setting $e^{-2\alpha\pi} = \sigma/\sigma - 1$, the equations (43) become

$$(44) \quad 0 = \sum_{j=1}^{\infty} A_j [\sigma \{x_{ij}^{(0)}(2\pi) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(2\pi) \mu^k\} - c_{ij}(\sigma - 1)] = 0, \\ i = 1, 2, \dots \infty.$$

To avoid the trivial case where the A_j are all zero, we must set the determinant

$$(45) \quad D(\sigma, \mu) = | [\sigma \{x_{ij}^{(0)}(2\pi) + \sum_{k=1}^{\infty} x_{ij}^{(k)}(\pi) \mu^k\} - c_{ij}(\sigma - 1)] | = 0,$$

which is a condition on the undetermined constant α . Since the $x_{ij}^{(0)} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k$ are the elements of a fundamental set, and $c_{ii} = 1$, it follows that $D(\sigma, \mu)$ converges absolutely for all finite σ 's. If the fundamental equation is satisfied when $\alpha = \alpha_0$, it is also satisfied when $\alpha = \alpha_0 + \nu\sqrt{-1}$, where ν is any integer; but all distinct solutions of the differential equations can be obtained by taking $\nu = 0$, since the ratios of the A_j are the same for $\nu = 0$ as for $\nu = p$. And for $\mu = 0$ the equation (45) reduces to

$$(46) \quad D_0 = | \sigma x_{ij}^{(0)}(2\pi) - c_{ij}(\sigma - 1) | = 0.$$

When $\mu = 0$, we have the case of constant coefficients which we treated in II. Now we shall determine the c_{ij} and the $\alpha^{(0)} = \lambda^{-1}$ of the solutions as we did there. But having the c_{ij} we might also determine the α of the solutions by means of (46). Since for given initial conditions the solution of the differential equations is unique, and since the initial conditions are the same in the two cases, the solutions obtained by the former method and the solution obtained by means of (46) are the same.

We recall that the c_{ij} are not all zero if and only if the characteristic equation has finite roots. Therefore for every root of the characteristic equation there is a corresponding root σ of (46); and conversely, for every root of (46) there is a corresponding root of the characteristic equation. There-

fore if the characteristic equation has no finite roots, then (46) has no finite roots, if the former has only a finite number of finite roots, then the latter has only a finite number of finite roots, then (46) has an infinite number of finite roots which yield independent solutions of the system (1).

Now since the solution (41) converges for all μ sufficiently small, including zero, the failure to find an α for $\mu = 0$ is not due to clumsy analytic methods, but shows that the system (1) has no solution of the form (41). From the theory of implicit functions we know that if (46) is satisfied when $\alpha = \alpha_0$, then we can solve for α as power series in μ , provided μ is sufficiently small. Therefore the existence of a root of the characteristic equation is both necessary and sufficient for the existence of a solution of the system (1) of the form (41), and the existence of an infinite number of roots of the characteristic equation is necessary and sufficient for the existence of a fundamental set of solutions of (1), each of the elements of which is of the form (41).

We shall study in detail the case that the characteristic equation has an infinite number of roots; and we shall use only those values of α obtained from (45) which for $\mu = 0$ reduce to the values of α obtained from the characteristic equation.

When $e^{-2a\pi} = 1$, the transformation $e^{-2a\pi} = \sigma/\sigma - 1$ can not be made. Then as in the case of constant coefficients when $\alpha = 0$, we can make no general statement about the solution, as the determinant of the A_i diverges then.

V

Solutions when the $\alpha_i^{(0)}$ are distinct and $\alpha_i^{(0)} - \alpha_j^{(0)} \not\equiv 0 \pmod{\sqrt{-1}}$.

The part of (45) which is independent of μ is

$$D_0 = | \sigma c_{ij} e^{2\alpha_j^{(0)}\pi} - c_{ij}(\sigma - 1) | = | c_{ij} | \prod_{j=1}^{\infty} \left(1 + \frac{1 - e^{2\alpha_j^{(0)}\pi}}{1 - e^{-2a\pi}} e^{-2a\pi} \right)$$

If (45) were an identity in μ , its roots would be the roots of (46), viz., $\alpha = \alpha_j^{(0)}$. Let us assume that we have the general case in which it is not an identity in μ , and set

$$(47) \quad \alpha = \alpha_k^{(0)} + \delta_k.$$

Then we get

$$(48) \quad D(\sigma, \mu) = D_0 + \mu F_k(\delta_k, \mu) = | c_{ij} | \left(1 - \frac{e^{-2\delta_k\pi} - e^{-2(\alpha_k^{(0)} + \delta_k)\pi}}{1 - e^{-2(\alpha_k^{(0)} + \delta_k)\pi}} \right) \prod_{j=1}^{\infty} \left(1 + \frac{1 - e^{-2\alpha_j^{(0)}\pi}}{1 - e^{-2(\alpha_k^{(0)} + \delta_k)\pi}} e^{-2(\alpha_k^{(0)} + \delta_k)\pi} \right) + \mu F_k(\delta_k, \mu) \quad j \neq k.$$

where $F_k(\delta_k, \mu)$ is a power series in μ and δ_k , converging for

$$|\delta_k| < \infty, |\mu| < \rho > 0.$$

Since by hypothesis no two of the $\alpha_1^{(0)}$ differ by an imaginary integer, the expansion of (48) as a power series in δ_k and μ contains a term in δ_k of the first degree and no term independent of both μ and δ_k . Therefore we know by the theory of implicit functions that (48) can be solved uniquely for δ_k as a power series of the form

$$(49) \quad \delta_k = \mu P_k(\mu),$$

which converges for $|\mu| > 0$ but sufficiently small.

Now we substitute this value of $\alpha = \alpha_k^{(0)} + \delta_k$ in (44), and get an infinite number of linear homogeneous equations for the A_j whose determinant converges and is zero, but the first minors of that determinant are not all zero, since by hypothesis, the roots of $D_0 = 0$ are all distinct and no two differ by an imaginary integer. Consequently these equations determine uniquely the ratios of the A_j as power series in μ , which converge for μ sufficiently small. On substituting these ratios in (42) we have the particular solution y_{ik} , $i = 1, 2, \dots, \infty$, expanded as a power series in μ . Hence we may write it

$$(50) \quad y_{ik} = \sum_{j=1}^{\infty} y_{ik}^{(j)}(t) \mu^j.$$

Since the periodicity conditions have been satisfied,

$$y_{ik}(t + 2\pi) - y_{ik}(t) = \sum_{j=1}^{\infty} [y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t)] \mu^j = 0,$$

for all μ sufficiently small and for all real t . Therefore

$$y_{ik}^{(j)}(t + 2\pi) - y_{ik}^{(j)}(t) = 0, \quad j = 0, 1, 2, \dots, \infty,$$

whence it follows that the $y_{ik}^{(j)}$, $j = 0, 1, 2, \dots, \infty$, are separately periodic.

A solution is found in a similar fashion for each $\alpha_j^{(0)}$.

VI

Solutions when no two of the $\alpha_j^{(0)}$ are equal but when $\alpha_2^{(0)} - \alpha_1^{(0)} \equiv 0 \pmod{\sqrt{-1}}$.

Suppose that when $\mu = 0$ the characteristic equation has two roots such that $\alpha_2^{(0)}$ and $\alpha_1^{(0)}$ differ by an imaginary integer and that none of the other $\alpha_j^{(0)}$ are congruent to $\alpha_1^{(0)} \pmod{\sqrt{-1}}$. Then we see from (45)

$$(51) \quad D(\sigma, \mu) = |c_{ij}| \left[1 - \frac{1 - e^{2a_1^{(0)}\pi}}{1 - e^{2(a_1^{(0)} + \delta_1)\pi}} \right]^2 \prod_{j=3}^{\infty} \left[1 - \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{2(a_1^{(0)} + \delta_1)\pi}} \right] + \mu F(\mu, \delta_1) = 0,$$

where as in (47) we have set $a = a_1^{(0)} + \delta_1$. The term of lowest degree in δ_1 alone is found by expanding the first bracket and turns out to be of the second degree. To get the terms in μ alone we suppress those involving δ_1 , after which we get a factor μ from each of the first two columns. So we see that in general the term of lowest degree in μ alone will be in this case of the second degree. Hence we have

$$(52) \quad D = |c_{ij}| \left[1 - \frac{1 - e^{2a_1^{(0)}\pi}}{1 - e^{2(a_1^{(0)} + \delta_1)\pi}} \right]^2 \prod_{j=3}^{\infty} \left[1 - \frac{1 - e^{2a_j^{(0)}\pi}}{1 - e^{2(a_1^{(0)} + \delta_1)\pi}} \right] + \delta_1 \mu F_1(\delta_1, \mu) + \delta^2 F_2(\delta_1, \mu) = 0.$$

In a similar manner if p of the $a_j^{(0)}$ are congruent to $a_1^{(0)} \pmod{\sqrt{-1}}$, then the term of lowest degree in δ_1 alone is of degree p , and in μ alone it is of at least the p th degree.

The problem of the form of the solution of (52) is one of implicit functions. Writing the first terms explicitly we have

$$\delta_1^2 + \kappa_{11}\delta_1\mu + \kappa_{02}\mu^2 + \text{terms of higher degree} = 0,$$

where $\kappa_{11}, \kappa_{02}, \dots$, are constants independent of δ_1 and μ . On factoring the quadratic terms we get

$$(53) \quad (\delta_1 - d_1\mu)(\delta_1 - d_2\mu) + \text{terms of higher degree} = 0.$$

If d_1 and d_2 are distinct, there are two solutions, and these have the form*

$$(54) \quad \delta_{11} = d_1\mu + \mu^2 P_1(\mu), \quad \delta_{12} = d_2\mu + \mu^2 P_2(\mu),$$

where P_1 and P_2 are power series which converge for μ sufficiently small. In this case the solutions are found as in V.

But if d_1 and d_2 are equal, the character of the solution is in general quite different and depends upon the terms of higher degree than the second. In general it will be a power series in $\pm\sqrt{\mu}$. This case we shall consider in detail.

We see from the form of (53) that the expansion of a_1 as a power series in $\sqrt{\mu}$ will contain no term in $\sqrt{\mu}$ to the first power, but will have the form

$$a_1 = a_1^{(0)} + 0\mu^{1/2} + a_1^{(1)}\mu + a_1^{(3/2)}\mu^{(3/2)} + \dots$$

Suppose that this expansion has been obtained from equation (45).

*Chrystal, "Algebra," Vol. 2, pp. 358 ff.

Then since α_1 is not a multiple zero of D , not all the first minors of D are zero when $\alpha = \alpha_1$. The ratios of the A_j will be determined from (44). If $\mu\Delta$ is a non-vanishing first minor corresponding to an element in the first column of D , it follows from the form of (45), remembering that we have $x_{ij}^{(0)}(t) = c_{ij}e^{\alpha_j^{(0)}t}$, that solving (44), we get

$$A_2 = \frac{\mu\Delta_2}{\mu\Delta} A_1, \quad A_j = \frac{\mu^2\Delta_j}{\mu\Delta} A_1, \quad j = 3, 4, \dots, \infty,$$

$$\text{where} \quad \Delta = \Delta^{(0)} + \Delta^{(1/2)}\mu^{1/2} + \Delta^{(1)}\mu + \dots, \\ \Delta_j = \Delta_j^{(0)} + \Delta_j^{(1/2)}\mu^{1/2} + \Delta_j^{(1)}\mu + \dots.$$

On substituting these series for the A_j in (42) we find that the y_{i_1} are developable as series of the form

$$y_{i_1} = y_{i_1}^{(0)} + y_{i_1}^{(1/2)}\mu^{1/2} + y_{i_1}^{(1)}\mu + \dots \quad i = 1, 2, \dots, \infty.$$

So we see that in general the y_{i_1} carry terms in $\sqrt{\mu}$ although the term in $\sqrt{\mu}$ is absent in the expansion for α_1 .

However, if all the first minors corresponding to the elements of the first column are zero, and if there is a first minor distinct from zero corresponding to the elements of the second column, the results are precisely the same. But suppose that all the first minors corresponding to the elements of both the first and second columns are zero. Then suppose that a first minor corresponding to an element of the k th column is not zero. Then it follows from the form of (45) that when $\alpha = \alpha_j$, it will carry the factor μ^2 ; and let this minor be denoted by $\mu^2\Delta$. Then solving (44) we get

$$A_1 = \frac{\mu\Delta_1}{\mu^2\Delta} A_k, \quad A_3 = \frac{\mu\Delta_2}{\mu^2\Delta} A_k, \quad A_j = \frac{\mu^2\Delta_j}{\mu^2\Delta} A_k, \quad j = 3, 4, \dots, \infty.$$

where $\Delta_1, \Delta_2, \Delta_j$, do not in general vanish when $\mu = 0$. It follows from the first two equations that A_k must carry μ as a factor, since the A_j are finite for $\mu = 0$. Hence in this case the y_{i_1} have the same form as before. Similarly the y_{i_2} have the same properties.

The solutions associated with $\alpha_3^{(0)}, \alpha_4^{(0)}, \dots$, are found as in the preceding case. If there are several groups of $\alpha_j^{(0)}$ in which these congruences exist the discussion must be made for each one separately.

VII

Solutions when $\alpha_1^{(0)}$ is a multiple root.

Now suppose that two of the $\alpha_j^{(0)}$ are equal and only two, and that there are none of the congruences treated in VI. Let us choose the notation so that $\alpha_1^{(0)} = \alpha_2^{(0)}$. From our work in the theory of linear differential equations

in infinitely many variables in constant coefficients and from IV, the solutions are of the form

$$(54) \quad x_i = \sum_{j=1}^{\infty} A_j [c_{ij} e^{a_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k] \quad \text{or}$$

$$(55) \quad x_i = A_1 [c_{i1} e^{a_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i1}^{(k)} \mu^k] + A_2 [(c_{i2} + t c_{i1}) e^{a_1^{(0)} t} + \sum_{k=1}^{\infty} x_{i2}^{(k)} \mu^k] \\ + \sum_{j=1}^{\infty} A_j [c_{ij} e^{a_j^{(0)} t} + \sum_{k=1}^{\infty} x_{ij}^{(k)} \mu^k].$$

After setting $e^{-2a\pi} = \sigma/\sigma - 1$, the fundamental equation becomes either

$$(56) \quad D(\sigma, \mu) = | [\sigma (c_{ij} e^{2a_j^{(0)} \pi} + \sum_{k=1}^{\infty} x_{ij}^{(k)} (2\pi) \mu^k) - c_{ij} (\sigma - 1)] | = 0 \quad \text{or}$$

$$(57) \quad D(\sigma, \mu) = | [\sigma \{ c_{i1} e^{2a_1^{(0)} \pi} + \sum_{k=1}^{\infty} x_{i1}^{(k)} (2\pi) \mu^k \} - c_{i1} (\sigma - 1)], \\ [\sigma \{ (c_{i2} + 2\pi c_{i1}) e^{2a_1^{(0)} \pi} + \sum_{k=1}^{\infty} x_{i2}^{(k)} (2\pi) \mu^k \} - c_{i2} (\sigma - 1)], \dots | = 0.$$

For $\mu = 0$ both of these, by the theory of infinite determinants, reduce to

$$(58) \quad D_0 = | c_{ij} | [1 - \sigma(1 - e^{2a_1^{(0)} \pi})]^2 \prod_{j=3}^{\infty} [1 - \sigma(1 - e^{2a_j^{(0)} \pi})].$$

As before we set $\alpha = \alpha_1^{(0)} + \delta_1$, and expand as a power series in δ_1 . Then we see that the term of lowest degree in δ_1 alone is of the second. When the determinant is of the form (56) with $\alpha_1^{(0)} = \alpha_2^{(0)}$, the term of lowest degree in μ alone is of the second in general. Then we have the same form as in VI. But if the determinant is of the form (57), the term in μ alone is in general of the first degree. In the former case we have a consideration similar to that in VI; in the latter case, in general the solutions for δ_1 are of the form

$$(59) \quad \delta_{11} = \mu^{1/2} P(\mu^{1/2}) \\ \delta_{12} = \mu^{1/2} P(-\mu^{1/2}),$$

where P is a power series in $\sqrt{\mu}$ and contains a term independent of μ . The discussion of the special cases is made just as in VI. On substituting these expansions for $\alpha = \alpha_1^{(0)} + \delta_1$ in (44) we solve for the A_j as power series in $\sqrt{\mu}$. These A_j substituted in (42) give y_{i1} and y_{i2} as power series in $\sqrt{\mu}$.

If p of the $\alpha_j^{(0)}$ are equal, then for these roots the expansions of D starts with δ_1^p as the term of lowest degree in δ_1 alone, and except in the special cases corresponding to those mentioned in the foregoing, the term in μ alone is of the first degree. Consequently in general for $\alpha_1^{(0)} = \alpha_2^{(0)} = \dots = \alpha_p^{(0)}$, we have

$$\delta_{ij} = \epsilon^j \mu^{1/p} P(\epsilon^j \mu^{1/p}), \quad j = 0, 1, \dots, p-1,$$

where ϵ is a p th root of unity.

VIII

Solutions when there are equalities and congruences among the $\alpha_j^{(0)}$.

Suppose that two of the $\alpha_j^{(0)}$, for example $\alpha_1^{(0)}$ and $\alpha_2^{(0)}$ are equal, and that a third one, say $\alpha_3^{(0)}$, differs from $\alpha_1^{(0)}$ by an imaginary integer. Furthermore we shall assume that there are no other equalities or congruences among the $\alpha_j^{(0)}$. Two cases arise here: (a) the solutions are of the form (54) with $\alpha_1^{(0)} = \alpha_2^{(0)}$; (b) the solutions are of the form (55)

Case (a). In this case we have

$$D_0 = |c_{ij}| [1 - \sigma(1 - e^{2\alpha_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1 - \sigma(1 - e^{2\alpha_j^{(0)}\pi})].$$

In setting $\alpha = \alpha_1^{(0)} + \delta_1$, we find that the term in D of lowest degree in δ_1 alone is of the third. To get the term in D of lowest degree in μ alone we set $\delta_1 = 0$. Then it becomes evident at once that each of the first columns carry μ as a factor, while the remaining ones do not. Consequently the term of lowest degree in μ is of the third degree at least. Furthermore since the first three columns vanish when $\mu = \delta_1 = 0$ there are no terms of lower degree than the third in δ_1 and μ . Hence in general we see D of the form

$$(60) \quad D = \delta_1^3 + \gamma_{21}\delta_1^2\mu + \gamma_{03}\mu^3 + \cdots = 0.$$

The problem is now one of implicit functions. The details of the special cases must be treated as they arise. However, we make the general statement that since the roots of the cubic terms of (60) set equal to zero are in general distinct, it follows from the theory of implicit functions that the three values of δ_1 are in general expandible in integral powers of μ .

Case (b). In this case we have

$$D_0 = [1 - \sigma(1 - e^{2\alpha_1^{(0)}\pi})]^3 \prod_{j=4}^{\infty} [1 - \sigma(1 - e^{2\alpha_j^{(0)}\pi})] = 0.$$

On introducing δ_1 as before, we find that the term of lowest degree in δ_1 alone is of the third degree. But when the terms involving μ are retained in D , only the first and third columns vanish when $\mu = \delta_1 = 0$, and consequently the expansion of D will contain a term in μ^2 alone. Furthermore, since the first and third columns vanish for $\mu = \delta_1 = 0$, there will be no terms of degree lower than the second in μ and δ_1 . Hence in general D has the form

$$(61) \quad D = \delta_1^3 + \gamma_{11}\delta_1\mu + \gamma_{02}\mu^2 + \cdots = 0.$$

In the general case in which γ_{11} and γ_{02} are not zero, there is one solution in integral powers of μ and two in powers of $\sqrt{\mu}$.

When the roots $\alpha_j^{(0)}$ have higher multiplicities and more congruences among them, we make a similar discussion.